Common Fixed-Point of Three Self-Mappings in Cone Metric Space

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Abstract

The aim of this paper is to prove a coincidence and common fixed-point theorem of three self-mappings satisfying contractive condition in cone metric space. Our findings improve and generalise certain well-known findings in the literature.

Keywords: Coincidence point, common fixed point, contractive condition.

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1 Introduction

Huang & Zhang [2] familiarized the idea of cone metric space and prove some fixed-point theorems for contractive type mappings in a normal cone metric space. Subsequently, some other authors [1, 3 to 6] studied the existence of fixed points of self-mappings satisfying a contractive type condition. Here, we obtain points of coincidence and common fixed points for three self-mappings satisfying condition (A) in a complete cone metric space.

The aim of this paper is to prove a coincidence and common fixed-point theorem of three self-mappings satisfying contractive type condition (A) in cone metric space.

2 Preliminaries

Definition 2.1 (see [1]**):** A subset P of a real Banach space E is called a cone if it has the following properties:

1. P is non-empty, closed and $P \neq \{0\}$
2. $0 \leq a, b \in \mathbb{R}$ and $\eta, v \in P \Longrightarrow a\eta + bv \in P$;
3. $\eta \in P$ and $-\eta \in P \Longrightarrow \eta = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Definition 2.2 (see [1]**):** For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by $\eta \leq v$ if and only if $\eta - v \in P$. We shall write $\eta < v$ if $\eta \leq v$ while $\eta \ll v$ stands for $v - \eta \in P^0$ where P^0 denotes the interior of P. The cone P is called normal if for some $K > 0$ for $\eta, v \in E$,

$$0 \leq \eta \leq v \Rightarrow \|\eta\| \leq K\|v\| \tag{2.1}$$

The least positive number K satisfying (2.1) is called the normal constant of P.

In the following, we always suppose that E is a real Banach space and P is a cone in E with $intP \neq 0$ and $\leq$ is a partial ordering with respect to P.

Proposition 2.3 (see [7]**):** Let P be a cone in a real Banach space E. If for $b \in P$ and $b \leq \gamma b$, for some $\gamma \in [0, 1)$ then $b = 0$.

Proposition 2.4 (see [7]**):** Let P be a cone in a real Banach space E with non-empty interior. If for $b \in E$ and $b \ll c$, for all $c \in P^0$, then $b = 0$.

Definition 2.5 (see [1]**):** Let X be a nonempty set. Suppose that the mapping $d: X \times X \rightarrow E$ satisfies

1. $0 \leq d(\eta, v), \forall\, \eta, v \in X$ and $d(\eta, v) = 0 \Leftrightarrow \eta = v$;
2. $d(\eta, v) = d(v, \eta), \forall\, \eta, v \in X$;
3. $d(\eta, v) \preccurlyeq d(\eta, w) + d(w, v), \forall\, \eta, v, w \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 2.6 (see [1]**):** Let $= \mathbb{R}^2$, $P = \{(\eta, v) \in E: \eta, v \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}$ and $d: X \times X \rightarrow E$ such that $d(\eta, v) = (|\eta - v|, \alpha|\eta - v|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is cone metric space.

Definition 2.7 (see [1]**):** Let $\{\eta_n\}$ be a sequence in X and $\eta \in X$. If for each $0 \ll c$, $\exists\, n_0 \in \mathbb{N}$ such that for all $n > n_0, d(\eta_n, \eta) \ll c$ then $\{\eta_n\}$ is said to be convergent (or $\{\eta_n\}$ converges) to η and η is called the limit of $\{\eta_n\}$. We denote this by $\lim_{n\to\infty} \eta_n = \eta$ or $\eta_n \rightarrow \eta$ as $n \rightarrow \infty$. If for each $0 \ll c\ \exists\ n_0 \in \mathbb{N}$ such that for all $n, m > n_0, d(\eta_n, \eta_m) \ll c$, then $\{\eta_n\}$ is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Definition 2.8 (see [1]**):** A pair (T, f) of self-mappings on X one said to be weakly compatible if they commute at their coincidence point i.e. $Tf\eta = fT\eta$ whenever $T\eta = f\eta$.

Definition 2.9 (see [1]**):** A point $v \in X$ is called a point of coincidence of T and f if ∃ a point $\eta \in X$ such that $v = T\eta = f\eta$.

Condition (A): Let (X, d) be a cone metric space, P be a normal cone with normal constant K and $g, f, T: X \to X$ are three self-mappings. Then g, f, T are said to satisfy condition (A) if

$$d(g\eta, fv) \leq ad(Tv, gv) + b[d(T\eta, g\eta) + d(Tv, fv)] + cd(T\eta, Tv)$$

for $\eta, v \in X$ where $0 \leq a, b, c < 1$ with $a + 2b + c < 1$.

Proposition 2.10 (see [6]**):** Let (X, d) be a cone metric space and P be a cone in a real Banach space E. If $\eta \leq v, v \ll w$ then $\eta \ll w$.

3 Main Results

Our main result as follows.

Theorem 3.1: Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose the mapping $g, f, T: X \to X$ satisfying condition (A). If $g(X) \cup f(X) \subseteq T(X)$ and $T(X)$ is a complete subspace of X, then g, f and T have a unique point of coincidence. Moreover, if $(g, T)$and (f, T) are weakly compatible, then g, f and T have a unique common fixed point.

Proof Let η_0 be arbitrary point in X.Choose a point η_1 in X such that $T\eta_0 = g\eta_1$. Similarly, choose a point η_2 in X such that $T\eta_2 = f\eta_1$.Continuing in this way choose η_k in X to obtain η_{k+1}in X such that

$$T\eta_{2k+1} = g\eta_{2k}, \quad T\eta_{2k+2} = f\eta_{2k+1}, \quad (k \geq 0)$$

Then,

$$d(T\eta_{2k+1}, T\eta_{2k+2}) = d(g\eta_{2k}, f\eta_{2k+1})$$

$$\leq ad(T\eta_{2k+1}, g\eta_{2k+1})$$

$$+b[d(T\eta_{2k}, g\eta_{2k}) + d(T\eta_{2k+1}, f\eta_{2k+1})] + cd(T\eta_{2k}, T\eta_{2k+1})$$

$$\leq ad(T\eta_{2k+1}, T\eta_{2k+2})$$

$$+b[d(T\eta_{2k}, T\eta_{2k+1}) + d(T\eta_{2k+1}, T\eta_{2k+2})] + cd(T\eta_{2k}, T\eta_{2k+1})$$

This $\Longrightarrow$

$$d(T\eta_{2k+1}, T\eta_{2k+2}) \leq \left[\frac{b+c}{1-(a+b)}\right] d(T\eta_{2k}, T\eta_{2k+1})$$

Similarly,

$$d(T\eta_{2k+2}, T\eta_{2k+3}) = d(g\eta_{2k+1}, f\eta_{2k+2})$$

$$\leq ad(T\eta_{2k+2}, g\eta_{2k+2})$$

$$+b[d(T\eta_{2k+1}, g\eta_{2k+1}) + d(T\eta_{2k+2}, f\eta_{2k+2})]$$

$$+cd(T\eta_{2k+1}, T\eta_{2k+2})$$

$$\leq ad(T\eta_{2k+2}, T\eta_{2k+3})$$

$$+b[d(T\eta_{2k+1}, T\eta_{2k+2}) + d(T\eta_{2k+2}, T\eta_{2k+3})]$$

$$+cd(T\eta_{2k+1}, T\eta_{2k+2})$$

This $\Longrightarrow$

$$d(T\eta_{2k+2}, T\eta_{2k+3}) \leq \left[\frac{b+c}{1-(a+b)}\right] d(T\eta_{2k+1}, T\eta_{2k+2})$$

Now by induction we arrive at

$$d(T\eta_{2k+1}, T\eta_{2k+2}) \leq \left[\frac{b+c}{1-(a+b)}\right] d(T\eta_{2k}, T\eta_{2k+1})$$

$$\leq \left[\frac{b+c}{1-(a+b)}\right]^2 d(T\eta_{2k-1}, T\eta_{2k})$$

$$\leq \ldots\ldots \leq \left[\frac{b+c}{1-(a+b)}\right]^k d(T\eta_0, T\eta_1)$$

And

$$d(T\eta_{2k+2},T\eta_{2k+3}) \leq \left[\frac{b+c}{1-(a+b)}\right] d(T\eta_{2k+1},T\eta_{2k+2})$$

$$\leq \left[\frac{b+c}{1-(a+b)}\right]^2 d(T\eta_{2k},T\eta_{2k+1})$$

$$\leq \ldots\ldots \leq \left[\frac{b+c}{1-(a+b)}\right]^{k+1} d(T\eta_0,T\eta_1)$$

for each $k \geq 0$.

Let $\lambda = \frac{b+c}{1-(a+b)}$, then $\lambda < 1$.

Hence in general we can write

$$d(T\eta_k,T\eta_{k+1}) \leq \lambda^k d(T\eta_0,T\eta_1)$$

Now for positive integer p, we have

$$d(T\eta_k,T\eta_{k+p}) \leq d(T\eta_k,T\eta_{k+1}) + d(T\eta_{k+2},T\eta_{k+3}) + \ldots\ldots + d(T\eta_{k+p-1},T\eta_{k+p})$$

$$\leq [\lambda^k + \lambda^{k+1} + \lambda^{k+2} + \ldots\ldots . + \lambda^{k+p-1}] d(T\eta_0,T\eta_1)$$

$$\leq \lambda^k \left[\frac{1-\lambda^{p-1}}{1-\lambda}\right] d(T\eta_0,T\eta_1)$$

$$\leq \lambda^k \left[\frac{1}{1-\lambda}\right] d(T\eta_0,T\eta_1)$$

Now for $c \in P^0$, $\exists\ r > 0$ such that $c - v \in P^0$ if $\|v\| < r$. Choose a positive integer n_0 such that for all $k \geq n_0$, $\left\|\frac{\lambda^k}{1-\lambda} d(T\eta_0,T\eta_1)\right\| < r$, which implies that,

$$c - \frac{\lambda^k}{1-\lambda} d(T\eta_0,T\eta_1) \in P^0 \text{ and } \frac{\lambda^k}{1-\lambda} d(T\eta_0,T\eta_1) - d(T\eta_k,T\eta_{k+p}) \in P^0$$

$k > N_c$ and for all P, by Proposition 1.10, $d(T\eta_k,T\eta_{k+p}) \ll c$ for all $k > N_c$ and for all P. Hence $\{T\eta_k\}$ is a Cauchy sequence in $T(X)$. Since $T(X)$ is complete, there exist $\omega, \varsigma \in X$ such that $T\eta_k \to \varsigma = T\omega$.

Since

$$d(T\omega, g\omega) \leq d(T\omega, T\eta_{2k}) + d(T\eta_{2k}, g\omega)$$

$$\leq d(\varsigma, T\eta_{2k}) + d(f\eta_{2k-1}, g\omega)$$

$$\leq d(\varsigma, T\eta_{2k}) + d(g\omega, f\eta_{2k-1})$$

$$\leq d(\varsigma, T\eta_{2k}) + ad(T\eta_{2k-1}, g\eta_{2k-1})$$

$$+b[d(T\omega, g\omega) + d(T\eta_{2k-1}, f\eta_{2k-1})] + cd(T\omega, T\eta_{2k-1})$$

$$= d(\varsigma, T\eta_{2k}) + ad(T\eta_{2k-1}, T\eta_{2k})$$

$$+b[d(T\omega, g\omega) + d(T\eta_{2k-1}, T\eta_{2k})] + cd(T\omega, T\eta_{2k-1})$$

This $\Longrightarrow$

$$d(T\omega, g\omega) \leq \frac{1}{1-b}\begin{bmatrix} d(\varsigma, T\eta_{2k}) + ad(T\eta_{2k-1}, T\eta_{2k}) \\ +bd(T\eta_{2k-1}, T\eta_{2k}) + cd(T\omega, T\eta_{2k-1}) \end{bmatrix}$$

Hence, it concludes that

$$\|d(T\omega, g\omega)\| \leq \frac{K}{1-b}\left\| \begin{matrix} d(\varsigma, T\eta_{2k}) + ad(T\eta_{2k-1}, T\eta_{2k}) \\ +bd(T\eta_{2k-1}, T\eta_{2k}) + cd(T\omega, T\eta_{2k-1}) \end{matrix} \right\|$$

where K is a normal constant. If $n \to \infty$, then we arrive at $\|d(T\omega, g\omega)\| = 0$. Hence $T\omega = g\omega$.

Similarly, by using the inequality

$$d(T\omega, f\omega) \leq d(T\omega, T\eta_{2k+1}) + d(T\eta_{2k+1}, f\omega)$$

We can show that $T\omega = f\omega$, implying that ς is a common point of coincidence of g, f and T; i.e. $\varsigma = T\omega = g\omega = f\omega$. Now we show that g, f and T have unique point of coincidence. For this, assume that there is another point ς^* in X, such that $\varsigma^* = T\omega^* = g\omega^* = f\omega^*$ for some $\omega^* \in X$. Now

$$d(\varsigma, \varsigma^*) = d(g\omega, f\omega^*)$$

$$\leq ad(T\omega^*, g\omega^*) + b[d(T\omega, g\omega) + d(T\omega^*, f\omega^*)]$$

$$+cd(T\omega, T\omega^*)$$

$$= ad(T\omega^*, T\omega^*) + b[d(T\omega, T\omega) + d(T\omega^*, T\omega^*)]$$

$$+cd(T\omega, T\omega^*)$$

The last inequality gives

$$d(\varsigma, \varsigma^*) \leq cd(\varsigma, \varsigma^*)$$

This is possible only when $\varsigma = \varsigma^*$.

If (g, T) and (f, T) are weakly compatible, then

$$g\varsigma = gT\omega = Tg\omega = T\varsigma \text{ and } f\varsigma = fT\omega = Tf\omega = T\varsigma$$

It implies that $g\varsigma = f\varsigma = T\varsigma = z$ (say). Hence, z is a point of coincidence of g, f and T and so $\varsigma = z$ by uniqueness. Thus ς is the unique common fixed point of g, f and T.

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