RC Factorization In Terms Of Reflexive Lower Triangular Matrix Over Complex Space

R.Indira[1*] and R.Sridevi[2]

[1,2]Department of Mathematics,
Anna University
Chennai-600025, Tamil Nadu, India.
Email: rindira2105@gmail.com
Email: rsridevi21@gmail.com

Abstract

In this article, we introduce the idea of reflexive lower triangular and reflexive upper triangular matrices. Also , we establish a new matrix product namely RC factorization for a given symmetric positive definite matrix in terms of reflexive lower triangular matrix. Numerical examples are provided.

Keywords: *Reflexive lower triangular matrix , Symmetric positive definite matrix,RC factorization.*

1. Introduction

Matrix factorization plays a vital role in linear algebra and other fields of Mathematics which has both scientific and engineering significance. Matrix decomposition or factorization is the process of splitting the given matrix into product of two or more which is more useful to reduce the computation involves in the matrix related problems.Various decompositions methods are available in the literature such as QR factorization, SVD method, LU decomposition, Cholesky decomposition and so on [1, 2 ,3]. Among all the above decompositions, Cholesky factorization method is simple and commonly used to solve system of linear equations and this factorization exists only for the symmetric and positive definite (SPD) matrix. The existence and computational algorithm of Cholesky method have been analysed by several authors such as Atkinson [1], Higham [4] and so on. Here, we are interested to present a new matrix product for a given SPD matrix which is analogous to Cholesky factorization and we call it as Reflexive Cholesky factorization or RC factorization.

In this paper, we introduce new class of triangular matrices namely reflexive lower triangular matrices and reflexive upper triangular matrices over complex space, as an extension of usual lower and upper triangular matrices. Also, we study some of its properties in section 2. In section 3, we establish the existence of new factorization namely Reflexive Cholesky factorization or RC factorization.

Any matrix $A \in \boldsymbol{C}^{n\times n}$ (the set of all $n \times n$complex matrices) is said to be symmetric if $A = A^T$ and is said to be hermitian if $A = A^*$, where A^T, A^*denote transpose, adjoint (conjugate transpose) of A respectively. $A \in R^{n\times n}$ (the set of all $n \times n$ real matrices) is said to be SPD matrix iff A is symmetric and $x^T Ax > 0$ for all non zero vector $x \in R^n$ $(set\ of\ n\ tuples\ over\ real\ numbers)$.

2. Reflexive lower triangular and Reflexive upper triangular matrices.

In this section, we define the idea of reflexive lower triangular and reflexive upper triangular matrices.

Definition 2.1 Any square matrix is said to be reflexive lower (ref- lower) triangular denoted by Ł if all the entries above the reflexive diagonal are zero. Equivalently, Ł of order n is defined as Ł $= [\boldsymbol{l_{ij}}]$, $i, j = 1\ to\ n$, of the form

$$\boldsymbol{l_{ij}} = \begin{cases} \boldsymbol{a_{ij}} & i+j>n, \\ 0 & \text{otherwise} \end{cases}.$$

.**Defintion 2.2** Any square matrix is said to be reflexive upper (ref-upper) triangular denoted by Ʉ if all the entries below the reflexive diagonal are zero.Equivalently Ʉ of order n is defined as Ʉ $= [\boldsymbol{u_{ij}}]$, $i, j = 1\ to\ n$, of the form $\boldsymbol{u_{ij}} =$

$$\begin{cases} \boldsymbol{a_{ij}} & i+j<n+2, \\ \boldsymbol{0} & \text{otherwise} \end{cases}.$$

Remark 2.3 The reflexive (secondary) diagonal matrix is a square matrix with zero entries except possibly on the reflexive diagonal (extends from the upper right corner to the lower left corner).

Example 2.4 $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 3 & 2 \\ 4 & 5i & 5 \end{pmatrix}$ is a ref-lower triangular matrix and

$B = \begin{pmatrix} 1 & 2 & 1 \\ i & 3 & 0 \\ 4 & 0 & 0 \end{pmatrix}$ is a ref- upper triangular matrix.

In the following theorem , we present some properties of ref- triangular matrices.

Theorem 2.5 Let $A, B \in \boldsymbol{C}^{n\times n}$. Then we have the following.

1. If A is ref- lower triangular then A^T is also a ref-lower triangular.

2. If A is ref- upper triangular thenA^T is also a ref- upper triangular.

3. If A is ref- lower (ref- upper) triangular and nonsingular if and only if none of the elements on its reflexive diagonal is zero.

4. If A is ref- lower triangular and B is ref- upper triangular then AB is lower triangular and BA is an upper triangular.

Proof:

1: Let $A = [a_{ij}]$. Since A is ref- lower triangular $a_{ij} = 0 \ for \ i + j \leq n$. If $A^T = [b_{ij}]$ then $b_{ij} = a_{ji}$ by definition of transpose. Thus $b_{ij} = 0 \, for \, i + j \leq n$.

Hence A^T is also a reflexive lower triangular.

2:.Proof is similar to that of 1.

The proof of 3,4 are obvious.

Remark 2.6 If A and B are ref-lower (ref- upper) triangular matrices then AB is not a ref-lower (ref- upper) triangular matrix respectively. This can be seen from the example

Example 2.7

$A = \begin{pmatrix} 0 & 1 \\ i & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2i \\ 1 & 1 \end{pmatrix}$ are ref- lower triangular matrices but

$AB = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is not a ref- lower triangular matrix. Similalrly, $A = \begin{pmatrix} 1 & 1 \\ i & 0 \end{pmatrix}$ and

$B = \begin{pmatrix} 2 & 2i \\ 1 & 0 \end{pmatrix}$ are reflexive upper triangular matrices but $AB = \begin{pmatrix} 3 & 2i \\ 2i & -2 \end{pmatrix}$ is not ref- upper triangular matrix.

3. Existence of RC factorization

In 1941, Crout discovered a decomposition method based on the fact that every matrix can be expressed as a product of a lower triangular matrix and an unit upper triangular matrix, Provided all the principal minors of a matrix are nonsingular. Later ,Cholesky used this factorization for a SPD matrix and he found that a matrix can be factorized into product of a lower triangular matrix and its transpose and it is known as Cholesky factorization [1].

From Theorem 2.5, it is clear that every matrix can not be expressed as a product of Ł and **Ʉ,** like crout's decomposion. Here we get a new factorization for a complex matrix in terms of Ł and its transpose and we call it as RC factorization. The following lemma found in [5] is required to obtain RC factorization.

Lemma 3.1 [p.34,5] Let$A \in R^{n\times n}$ be a SPD matrix . Partitioning A as

$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with $a_{11} \in R,\ A_{12} \in R^{1\times n-1},\ A_{21} = {A_{12}}^T\ and\ A_{22} \in R^{n-1\times n-1}$,

and define the Schur complement of A with respect to a_{11}, as

$A/_{a_{11}} = A_{22} - \frac{1}{a_{11}} A_{21}\, A_{12}$. Then $A/_{a_{11}}$ is also SPD.

Now, we prove our main result, the existence of RC factorization in the following theorem for the real case .

Theorem 3.2 An invertible matrix $A \in R^{n\times n}$, has RC factorization $A = Ł\,Ł^T$ if and only if A is symmetric and positive definite, where Ł is a ref-lower triangular matrix of order n.

Proof:

Suppose $A = Ł\,Ł^T$. Then $A^T = (Ł\,Ł^T)^T = (Ł^T)^T Ł^T = Ł\,Ł^T = A$, which implies A is symmetric. To show A is positive definite, let $x \in R^n - \{0\}$, then
$x^T A x = x^T Ł\,Ł^T x = (Ł^T x)^T\,(Ł^T x) = \|Ł^T x\|_2{}^2$.

Since A is assumed to be an invertible so is the matrix Ł and also $Ł^T$it follows from,

$0 \neq detA =$ det $(Ł\,Ł^T) = (\det Ł)^2$. Since $x \neq 0$ *, implies* $Ł^T x \neq 0$ and consequently, $\left\|Ł^T x\right\|_2 > 0$ which implies $x^T A x > 0$. Thus A is symmetric and positive definite .

Conversely, we have to prove every symmetric and positive definite matrix has RC factorization.

Proof by an induction.

When$n = 1$, the result is true for a 1×1 matrix $A = a_{11}$. Since A is SPD, $a_{11} > 0$. Define the RC factor $Ł = \sqrt{a_{11}}$.

Assume that the result is true for a SPD matrix $A \in R^{n-1\times n-1}$. We will show that the result is true for a SPD matrix $A \in R^{n\times n}$.

For let $A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ With $a_{11} \in R$; $A_{12} \in R^{1\times n-1}$; $A_{21} = A_{12}^T$ *and* $A_{22} \in R^{n-1\times n-1}$. The Schur-complement of A with respect to a_{11},

$A/_{a_{11}} = A_{22} - \frac{1}{a_{11}} A_{21}\,A_{12} \in R^{n-1\times n-1}$ is SPD by Lemma 3.1.

Using the induction hypothesis, we therefore conclude that $A/_{a_{11}}$ has RC factorization. That is $A/_{a_{11}} = Ł_1 Ł_1{}^T$ for some ref - lower triangular matrix $Ł_1$ of order $n - 1$.

Define $Ł = \begin{pmatrix} 0 & \sqrt{a_{11}} \\ Ł_1 & \frac{A_{21}}{\sqrt{a_{11}}} \end{pmatrix}$. Then

$$Ł\,Ł^T = \begin{pmatrix} 0 & \sqrt{a_{11}} \\ Ł_1 & \frac{A_{21}}{\sqrt{a_{11}}} \end{pmatrix} \begin{pmatrix} 0 & Ł_1{}^T \\ \sqrt{a_{11}} & \frac{A_{12}}{\sqrt{a_{11}}} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & Ł_1Ł_1{}^T + \frac{1}{a_{11}} A_{21}A_{12} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A/_{a_{11}} + \frac{1}{a_{11}} A_{21}A_{12} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A.$$

Thus we have constructed a RC factorization of A, and Ł is a RC factor, which concludes the induction step.

The above theorem can be illustrated in the following example.

Example 3.3

Consider a matrix $A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 5 \end{bmatrix}$.

Here A is a SPD matrix with $a_{11} = 4$, $A_{12} = [2 \quad -2]$

$A_{21} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $\qquad A_{22} = \begin{bmatrix} 10 & 2 \\ 2 & 5 \end{bmatrix}$.

Also ,

$$A/_{a_{11}} = A_{22} - \frac{1}{a_{11}} A_{21}A_{12} = \begin{bmatrix} 10 & 2 \\ 2 & 5 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 \\ -2 \end{bmatrix} [2 \quad -2]$$

$$= \begin{bmatrix} 9 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{3} \\ 3 & 1 \end{bmatrix}.$$

$$Ł\,Ł^T = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 1 \\ \sqrt{3} & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \sqrt{3} \\ 0 & 3 & 1 \\ 2 & 1 & -1 \end{bmatrix} = A$$

Remark 3.4 The factorization obtained in Theorem 3.2 is unique and analogous to Cholesky factorization. So, we named it as Reflexive Cholesky factorization or RC factorization.

The above Theorem 3.2 can also be extended for complex matrices. Now, we state RC factorization for a complex matrix A in the following .

Theorem 3.5 An invertible matrix $A \in \boldsymbol{C}^{n \times n}$ has RC factorization $A = \text{Ł}\,\text{Ł}^*$ if and only if A is hermitian and positive definite, where Ł is a ref-lower triangular matrix of order n.

Proof:

Using the same proof as above given in Theorem 3.2, replacing each transpose by adjoint we can prove that a complex invertible matrix A has RC factorization if and only if A is hermitian and positive definite.

Conclusion

In this paper, we have introduced a class of reflexive triangular matrices. As an application of reflexive lower triangular matrices , we have obtained RC factorization of a symmetric positive definite matrix. RC factorization can be used to solve linear system equations similar to Cholesky factorization.

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