# Effect of Customer Impatience in an *M/M/1* Queue with Bernoulli Scheduled Working Vacation and Interruption

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## Abstract

*Vacation queuing models have diverse applications in many practical scenarios. Queuing models wherein the person providing service is temporarily unavailable to provide service are classified as vacation models. A single server queueing model is studied in which the server takes a short break when there are no waiting customers to avail service. However, to avoid the loss of customers who arrive during the vacation period, it is assumed that the service continues at a slower rate by an alternate server. Such a slow service, in turn, results in the act of customers leaving the queue because of extended waiting time. Furthermore, after completing each service, the server might interrupt the vacation with some positive probability. Explicit form of analytical results for the system size probabilities during the server's busy and vacation period are obtained. The numerical study provides insights into the theoretical results so obtained.*

***Keywords:*** *Continued Fractions, Modified Bessel Functions, Vacation Interruption, Laplace Transforms, Confluent Hypergeometric Function.*

## 1. Introduction

Queueing models subject to different vacation strategies attracts many kinds of research owing to its broad applicability in several practical scenarios. For an in-depth understanding of vacation queuing models, the readers may refer to the book by Tian and Zhang [9] and the survey by Ke et al. [2]. Servi and Finn [6] initiated the idea of a server who continues to provide service at a slower rate in a single server queue with an application to the analysis of a WDM network. Liu et al. [4] developed a stochastic decomposition result of Servi and Finn's model. Following that, several authors studied vacation queueing models subject to different vacation policies. Zhang and Shi [12] analysed a single server queue with multiple vacations and interruption wherein they derived the stationary distribution of the queue length.

Nevertheless, another factor that primarily affects the queueing behaviour is the impatient behaviour of the customer. Kim and Kim [3] obtains the system size distribution for a queue where the rate of service obeys a continuous random process. The probability generating function of a multiple server queueing model subject to impatient behaviour of the arriving customers was obtained by Yu and Liu [11]. However, most of the results available in the literature pertains to the steady-state analysis. The transient analysis helps to investigate the model at any arbitrary epoch. The current model focuses on the time-dependent study of an *M/M/1* queueing model with customer impatience, Bernoulli scheduled vacation and interruption of vacation. Veena Goswami [10] dealt with the stationary analysis of this model.

## 2. Mathematical Formulation

Consider an *M/M/1* queueing model. At the service completion instant, if no more customers require service, the server takes a short break of fixed duration (termed as a period of vacation). However, arrivals continue to join the queue during vacation times and to avoid losing potential customers; the service is assumed to continue at a slower rate. This reduced rate of service will, in turn, result in customer impatience because of the extended waiting time. Furthermore, during the vacation duration of the server, after completing each service, the server will continue to be in the vacation mode or interrupt the vacation with probability $q$ and $1-q$ respectively. Such a model is appropriate in many human involved systems.

The following notations are used in our model:

$\lambda$ −Poisson rate of arrival
$\mu$ − exponential rate of service time
$\phi$ −exponential rate ofvacation time
$\eta$ −exponential rate of service time during vacation duration
$\xi$ −exponential rate of customer impatience
$N(t)$ −system size at time $t$
$S(t)$ −state of the server at time $t$ such that

$$S(t) = \begin{cases} 1, \text{during the server's normal busy period.} \\ 0, \text{during the server's working vacation period.} \end{cases}$$

$S$ − State space of the process, $\{N(t), S(t)\}$ given by

$$S = \{(0,0) \cup (n,j); n = 1,2,3,\ldots; j = 0,1\}$$

$\mathcal{P}_{0,0}(t) = Prob\ (N(t) = 0, S(t) = 0)$,
$\mathcal{P}_{n,j}(t) = Prob\ (N(t) = n, S(t) = j), \quad n = 1,2,3,..,j = 0,1.$

Figure 1 illustrates the possible transitions that take place among the various states along with the corresponding rate at which they happen.

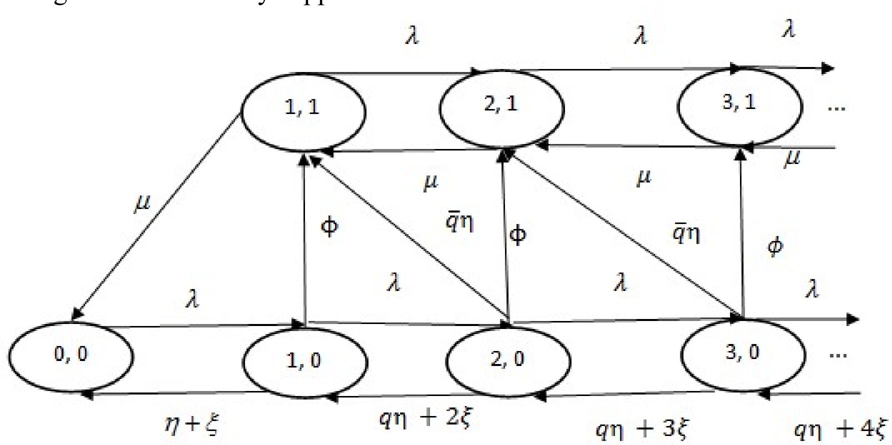


**Figure 1.** State Transition Diagram

The infinite system of equations which regulates the Markov process are presented below:

$$\mathcal{P}'_{0,0}(t) = -\lambda\mathcal{P}_{0,0}(t) + \mu\mathcal{P}_{1,1}(t) + (\eta + \xi)\mathcal{P}_{1,0}(t), \qquad (2.1)$$

$$\mathcal{P}'_{1,0}(t) = \lambda\mathcal{P}_{0,0}(t) - (\lambda + \phi + (\eta + \xi))\mathcal{P}_{1,0}(t) + (q\eta + 2\xi)\mathcal{P}_{2,0}(t), \qquad (2.2)$$

$$\mathcal{P}'_{n,0}(t) = \lambda\mathcal{P}_{n-1,0}(t) - (\lambda + \phi + (\eta + n\xi))\mathcal{P}_{n,0}(t) + (q\eta + (n+1)\xi)\mathcal{P}_{n+1,0}(t),$$

$$n = 2,3,4 \ldots \qquad (2.3)$$

$$\mathcal{P}'_{1,1}(t) = \phi\mathcal{P}_{1,0}(t) - (\lambda + \mu)\mathcal{P}_{1,1}(t) + \bar{q}\eta\mathcal{P}_{2,0}(t) + \mu\mathcal{P}_{2,1}(t), \qquad (2.4)$$

$$\mathcal{P}'_{n,1}(t) = \lambda\mathcal{P}_{n-1,1}(t) - (\lambda + \mu)\mathcal{P}_{n,1}(t) + \mu\mathcal{P}_{n+1,1}(t) + \phi\mathcal{P}_{n,0}(t) + \bar{q}\eta\mathcal{P}_{n+1,0}(t),$$

$$n = 2,3,4 \ldots \qquad (2.5)$$

with $\mathcal{P}_{0,0}(0) = 1$ and $\mathcal{P}_{n,j}(0) = 0$ for all other possible values of *n* and *j*.

## 3. Time-Dependent Analysis

This section attempts to find a solution to the infinite system of governing equations and hence obtain the transient probabilities, which aids in the thorough analysis of various measures of effectiveness associated with the model. In the process, we make use of several exciting techniques like continued fractions and generating functions.

Laplace Transform in equation (2.1) yields

$$(s + \lambda)\hat{\mathcal{P}}_{0,0}(s) = 1 + (\eta + \xi)\hat{\mathcal{P}}_{1,0}(s) + \mu\hat{\mathcal{P}}_{1,1}(s)$$

Hence

$$\hat{\mathcal{P}}_{0,0}(s) = \frac{1}{(s + \lambda)} + \frac{\eta + \xi}{s + \lambda}\hat{\mathcal{P}}_{1,0}(s) + \frac{\mu}{s + \lambda}\hat{\mathcal{P}}_{1,1}(s) \qquad (3.1)$$

Similarly, Laplace transform of equations (2.3) results in

$$s\hat{\mathcal{P}}_{n,0}(s) - \mathcal{P}_{n,0}(0) = -\big(\lambda + \phi + (\eta + n\xi)\big)\hat{\mathcal{P}}_{2,n}(s) + \lambda\hat{\mathcal{P}}_{n-1,0}(s) + (q\eta + (n+1)\xi)\hat{\mathcal{P}}_{n+1,0}(s).$$

Using the initial conditions in the above equation results in

$$(s + \lambda + \phi + (\eta + n\xi))\hat{\mathcal{P}}_{n,0}(s) = \lambda\hat{\mathcal{P}}_{n-1,0}(s) + (q\eta + (n+1)\xi)\hat{\mathcal{P}}_{n+1,0}(s)$$

The above equation is expressed as

$$\frac{\hat{\mathcal{P}}_{n,0}(s)}{\hat{\mathcal{P}}_{n-1,0}(s)} = \frac{\lambda}{(s + \lambda + \phi + (\eta + n\xi)) - (q\eta + (n+1)\xi)\dfrac{\hat{\mathcal{P}}_{n+1,0}(s)}{\hat{\mathcal{P}}_{n,0}(s)}}$$

which further leads to the continued fraction given by

$$\frac{\hat{\mathcal{P}}_{n,0}(s)}{\hat{\mathcal{P}}_{n-1,0}(s)} = \cfrac{\lambda}{\big(s + \lambda + \phi + (\eta + n\xi)\big) - \cfrac{\lambda(q\eta + (n+1)\xi}{\big(s + \lambda + \phi + (\eta + (n+1)\xi)\big) - \cfrac{\lambda(q\eta + (n+2)\xi}{(s + \lambda + \phi + (\eta + (n+2)\xi)} - \cdots}}$$

Below, we present an interesting identity relating continued fractions and hypergeometric series taken from Lorentzen and Waadeland [[9], (4.1.5), p.573] and is given by

$$\frac{{}_1F_1(a+1;c+1;z)}{{}_1F_1(a;c;z)}=\frac{c}{c-z+}\frac{(a+1)z}{c-z+1+}\frac{(a+2)z}{c-z+2+}\cdots,$$

where $1F1(a;c;z)$denotes the confluent hypergeometric function having a series representation given by

$${}_1F_1(a;c;z)=1+\frac{a}{c}\frac{z}{1!}+\frac{a(a+1)}{c(c+1)}\frac{z^2}{2!}+\cdots==\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{z^k}{k!},$$

Using the above fact, we observe that for $n=1,2,3\ldots,$

$$\frac{\hat{\mathcal{P}}_{n,0}(s)}{\hat{\mathcal{P}}_{n-1,0}(s)}=\frac{\lambda}{\xi\left(\frac{s+\phi+\eta}{\xi}+n\right)}\frac{1F1\left(\frac{q\eta}{\xi}+n+1;\frac{s+\phi+\eta}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{1F1\left(\frac{q\eta}{\xi}+n;\frac{s+\phi+\eta}{\xi}+n;-\frac{\lambda}{\xi}\right)},$$

Hence, we recursively get

$$\hat{\mathcal{P}}_{n,0}(s)=\hat{\psi}_n(s)\hat{\mathcal{P}}_{0,0}(s) \tag{3.2}$$

where

$$\hat{\psi}_n(s)=\left(\frac{\lambda}{\xi}\right)^n\frac{1}{\prod_{i=1}^{n}\left(\frac{s+\phi+\eta}{\xi}+i\right)}\frac{1F1\left(\frac{q\eta}{\xi}+n+1;\frac{s+\phi+\eta}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{1F1\left(\frac{q\eta}{\xi}+1;\frac{s+\phi+\eta}{\xi}+1;-\frac{\lambda}{\xi}\right)}.$$

Specifically, when $n=1$, equation (3.2) becomes

$$\hat{\mathcal{P}}_{1,0}(s)=\hat{\psi}_1(s)\hat{\mathcal{P}}_{0,0}(s) \tag{3.3}$$

Substituting the equation (3.3) in the equation (3.1) results in

$$\hat{\mathcal{P}}_{0,0}(s)=\frac{1}{s+\lambda}+\frac{\eta+\xi}{s+\lambda}\left(\hat{\psi}_1(s)\hat{\mathcal{P}}_{0,0}(s)\right)+\frac{\mu}{s+\lambda}\hat{\mathcal{P}}_{1,1}(s). \tag{3.4}$$

After some algebra, we get

$$\hat{\mathcal{P}}_{0,0}(s)=\left(1+\mu\hat{\mathcal{P}}_{1,1}(s)\right)\sum_{r=0}^{\infty}\frac{(\eta+\xi)^r\left[\hat{\psi}_1(s)\right]^r}{(s+\lambda)^{r+1}}. \tag{3.5}$$

Using the equation (3.5) in equation (3.2) yields

$$\hat{\mathcal{P}}_{n,0}(s)=\left(1+\mu\hat{\mathcal{P}}_{1,1}(s)\right)\hat{\psi}_n(s)\sum_{r=0}^{\infty}\frac{(\eta+\xi)^r\left[\hat{\psi}_1(s)\right]^r}{(s+\lambda)^{r+1}},n=1,2,3\ldots \tag{3.6}$$

Taking Laplace inverse for equation (3.5) and equation (3.6) results in

$$\mathcal{P}_{0,0}(t) = \left(\delta(t) + \mu\mathcal{P}_{1,1}(t)\right) * \sum_{r=0}^{\infty}\left\{e^{-\lambda t}\frac{\left((\eta+\xi)t\right)^r}{r!} * \psi_1[(t)]^{*r}\right\},$$

and

$$\mathcal{P}_{n,0}(t) = \psi_n(t) * \mathcal{P}_{0,0}(t), n = 1,2,3, \ldots \tag{3.7}$$

where $\delta(t)$ is the Kronecker delta function and $\psi_n(t)$ for all values of *n* are derived in the Appendix. Having found $\mathcal{P}_{n,0}(t)$ for all possible values of *n* in terms of $\mathcal{P}_{1,1}(t)$, we present below the method to determine $\mathcal{P}_{n,1}(t)$ for all possible *n*.

***Evaluation of $\mathcal{P}_{n,1}(t), n = 1, 2, 3 \ldots$***

Let

$$H(z,t) = \sum_{n=1}^{\infty}\mathcal{P}_{n,1}(t)z^n.$$

Then,

$$\frac{\partial H(z,t)}{\partial t} = \sum_{n=1}^{\infty}\mathcal{P}'_{n,1}(t)z^n.$$

Multiplying equation (2.5) by $z^n$ and adding it over all possible values of *n* results in

$$\sum_{n=2}^{\infty}\mathcal{P}'_{n,1}(t)z^n = -(\lambda+\mu)\sum_{n=2}^{\infty}\mathcal{P}_{n,1}(t)z^n + \mu\sum_{n=2}^{\infty}\mathcal{P}_{n+1,1}(t)z^n + \lambda\sum_{n=2}^{\infty}\mathcal{P}_{n-1,1}(t)z^n + \phi\sum_{n=2}^{\infty}\mathcal{P}_{n,0}(t)z^n + \bar{q}\eta\sum_{n=2}^{\infty}\mathcal{P}_{n+1,0}(t)\,z^n$$

Using the definition of $H(z,t)$ and its partial derivative yields

$$\frac{\partial H(z,t)}{\partial t} - \left(-(\lambda+\mu) + \frac{\mu}{z} + \lambda z\right)H(z,t) = \left(\phi + \frac{\bar{q}\eta}{z}\right)\sum_{n=1}^{\infty}z^n\mathcal{P}_{n,0}(t) - \mu\mathcal{P}_{1,1}(t) - \bar{q}\eta\mathcal{P}_{1,0}(t). \tag{3.8}$$

As we integrate equation (3.8), we get

$$H(z,t) = \phi\int_0^t\sum_{n=1}^{\infty}\mathcal{P}_{n,0}(y)z^n\,e^{-(\lambda+\mu)(t-y)}e^{\left(\frac{\mu}{z}+\lambda z\right)(t-y)}\,dy + \bar{q}\eta\int_0^t\sum_{n=1}^{\infty}\mathcal{P}_{n+1,0}(t)z^n\,e^{-(\lambda+\mu)(t-y)}e^{\left(\frac{\mu}{z}+\lambda z\right)(t-y)}dy - \mu\int_0^t\mathcal{P}_{1,1}(y)e^{-(\lambda+\mu)(t-y)}e^{\left(\frac{\mu}{z}+\lambda z\right)(t-y)}dy \tag{3.9}$$

With $\alpha = 2\sqrt{\lambda\mu}$ and$\beta = \sqrt{\frac{\lambda}{\mu}}$, the generating function of the modified Bessel function of order *n* represented by $I_n(.)$ is given by

$$exp\left(\frac{\mu t}{z} + \lambda z t\right) = \sum_{n=-\infty}^{\infty}(\beta z)^n I_n(\alpha t).$$

Extracting the numerical constant in $z^n$ from equation (3.9) for positive values of $n$ results in

$$\mathcal{P}_{n,1}(t) = \phi \int_0^t \sum_{i=1}^{\infty} \mathcal{P}_{i,0}(y)\beta^{n-i} I_{n-i}\big(\alpha(t-y)\big) e^{-(\lambda+\mu)(t-y)}\, dy$$
$$+ \bar{q}\eta \int_0^t \sum_{i=1}^{\infty} \mathcal{P}_{i+1,0}(y)\beta^{n-i} I_{n-i}\big(\alpha(t-y)\big) e^{-(\lambda+\mu)(t-y)}\, dy$$
$$- \mu \int_0^t \mathcal{P}_{1,1}(y)\beta^{n} I_{n}\big(\alpha(t-y)\big) e^{-(\lambda+\mu)(t-y)} dy. \quad (3.10)$$

In a similar way, extracting the numerical constant in $z^{-n}$ from equation (3.10) for negative values of $'n'$ results in

$$0 = \phi \int_0^t \sum_{i=1}^{\infty} \mathcal{P}_{i,0}(y)\beta^{-n-i} I_{-n+i}\big(\alpha(t-y)\big) e^{-(\lambda+\mu)(t-y)}\, dy$$
$$+ \bar{q}\eta \int_0^t \sum_{i=1}^{\infty} \mathcal{P}_{i+1,0}(y)\beta^{-n-i} I_{-n+i}\big(\alpha(t-y)\big) e^{-(\lambda+\mu)(t-y)}\, dy$$
$$- \mu \int_0^t \mathcal{P}_{1,1}(y)\beta^{-n} I_{-n}\big(\alpha(t-y)\big) e^{-(\lambda+\mu)(t-y)} dy.$$

Multiplying the above equation by $\beta^{2n}$ and exploiting the fact $I_{-n}(t) = I_n(t)$, we get

$$0 = \phi \int_0^t \sum_{i=1}^{\infty} \mathcal{P}_{i,0}(y)\beta^{n-i} I_{n-i}\big(\alpha(t-y)\big) e^{-(\lambda+\mu)(t-y)}\, dy$$
$$+ \bar{q}\eta \int_0^t \sum_{i=1}^{\infty} \mathcal{P}_{i+1,0}(y)\beta^{n-i} I_{n+i}\big(\alpha(t-y)\big) e^{-(\lambda+\mu)(t-y)}\, dy$$
$$- \mu \int_0^t \mathcal{P}_{1,1}(y)\beta^{n} I_{n}\big(\alpha(t-y)\big) e^{-(\lambda+\mu)(t-y)} dy. \quad (3.11)$$

Now, the subtraction of equation (3.10) from equation (3.11) leads to

$$\mathcal{P}_{n,1}(t) = \int_0^t \sum_{i=1}^{\infty} \big[\phi\mathcal{P}_{i,0}(y) + \bar{q}\eta\mathcal{P}_{i+1,0}(y)\big]\beta^{n-i}$$
$$\Big(I_{n-i}\big(\alpha(t-y)\big) - I_{n+i}\big(\alpha(t-y)\big)\Big) e^{-(\lambda+\mu)(t-y)} dy \quad (3.12)$$

for $n = 1,2,3,\ldots$. Thus $\mathcal{P}_{n,1}(t)$ is expressed in terms of $\mathcal{P}_{n,0}(t)$. To obtain $\mathcal{P}_{1,1}(t)$ explicitly, let us substitute $n = 1$ in equation (3.12) and get

$$\mathcal{P}_{1,1}(t) = \int_0^t \sum_{i=1}^{\infty} \big[\phi\mathcal{P}_{i,0}(y) + \bar{q}\eta\mathcal{P}_{i+1,0}(y)\big]\beta^{1-i}$$
$$\Big(I_{1-i}\big(\alpha(t-y)\big) - I_{1+i}\big(\alpha(t-y)\big)\Big) e^{-(\lambda+\mu)(t-y)} dy \quad (3.13)$$

Using the property $I_{m-1}(t) - I_{m+1}(t) = \frac{2km(t)}{t}$ and $I_{1-m}(t) = I_{m-1}(t)$, we get

$$\mathcal{P}_{1,1}(t) = \int_0^t \sum_{i=1}^{\infty} \big[\phi\mathcal{P}_{i,0}(y) + \bar{q}\eta\mathcal{P}_{i+1,0}(y)\big]\beta^{1-i} \frac{2iI_i(\alpha(t-y))}{\alpha(t-y)} e^{-(\lambda+\mu)(t-y)}\, dy. \quad (3.14)$$

Laplace transform of equation (3.14) yields

$$\hat{\mathcal{P}}_{1,1}(s) = \sum_{i=1}^{\infty} \big[\phi\hat{\mathcal{P}}_{i,0}(s) + \bar{q}\eta\hat{\mathcal{P}}_{i+1,0}(s)\big]\beta^{1-i} \frac{1}{\alpha^{-i+1}\Big(p + \sqrt{p^2 - \alpha^2}\Big)^{i}}$$

where $p = s + \lambda + \mu$. It can also be written as

$$\hat{\mathcal{P}}_{1,1}(s) = \frac{1}{\mu}\sum_{i=1}^{\infty}\left[\phi\hat{\mathcal{P}}_{i,0}(s) + \bar{q}\eta\hat{\mathcal{P}}_{i+1,0}(s)\right]\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^i \tag{3.15}$$

Substituting for $\hat{P}_{i,0}(s)$ and $\hat{P}_{i+1,0}(s)$ in equation (3.15) leads to

$$\hat{\mathcal{P}}_{1,1}(s) = \frac{1}{\mu}\left(\sum_{i=1}^{\infty}\left[\phi\left(1 + \mu\hat{\mathcal{P}}_{1,1}(s)\right)\hat{\psi}_i(s)\sum_{r=0}^{\infty}\frac{(\eta+\xi)^r\left[\hat{\psi}_1(s)\right]^r}{(s+\lambda)^{r+1}} + \bar{q}\eta\left(1 + \mu\hat{\mathcal{P}}_{1,1}(s)\right)\hat{\psi}_{i+1}(s)\sum_{r=0}^{\infty}\frac{(\eta+\xi)^r\left[\hat{\psi}_1(s)\right]^r}{(s+\lambda)^{r+1}}\right]\right)\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^i,$$

and hence

$$\hat{\mathcal{P}}_{1,1}(s) = \frac{1}{\mu}\sum_{r=0}^{\infty}\frac{(\eta+\xi)^r\left[\hat{\psi}_1(s)\right]^r}{(s+\lambda)^{r+1}}\sum_{i=1}^{\infty}\left(\phi\hat{\psi}_i(s) + \bar{q}\eta\hat{\psi}_{i+1}(s)\right)\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^i \sum_{m=0}^{\infty}\left(\sum_{r=0}^{\infty}\frac{(\eta+\xi)^r\left[\hat{\psi}_1(s)\right]^r}{(s+\lambda)^{r+1}}\sum_{i=1}^{\infty}\left(\phi\hat{\psi}_i(s) + \bar{q}\eta\hat{\psi}_{i+1}(s)\right)\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^i\right)^{\mathrm{m}},$$

which can be expressed as

$$\hat{\mathcal{P}}_{1,1}(s) = \frac{1}{\mu}\sum_{m=0}^{\infty}(\hat{w}(\mathrm{s}))^{\mathrm{m+1}}, \tag{3.16}$$

where

$$\hat{w}(\mathrm{s}) = \sum_{r=0}^{\infty}\frac{(\eta+\xi)^r\left[\hat{\psi}_1(s)\right]^r}{(s+\lambda)^{r+1}}\sum_{\mathrm{i}=1}^{\infty}\left(\phi\hat{\psi}_i(s) + \bar{q}\eta\hat{\psi}_{i+1}(s)\right)\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^i.$$

Taking Laplace inverse for equation (3.16) leads to

$$\mathcal{P}_{1,1}(t) = \frac{1}{\mu}\sum_{m=0}^{\infty}(w(t))^{*(\mathrm{m+1})},$$

where

$$w(t) = \left\{\sum_{r=0}^{\infty}e^{-\lambda t}\frac{\left((\eta+\xi)t\right)^r}{r!} * [\psi_1(t)]^{*r}\right\} * \sum_{\mathrm{i}=1}^{\infty}\left(\phi\psi_i(t) + \bar{q}\eta\psi_{i+1}(t)\right) * \frac{I_i(\alpha t)}{\beta^i t}. \tag{3.17}$$

In summary, $\mathcal{P}_{n,j}(t)$ for varying values of *n* and *j* are

$$\mathcal{P}_{1,1}(t) = \frac{1}{\mu}\sum_{m=0}^{\infty}(w(t))^{*(\mathrm{m+1})},$$

where $w(t)$ is given by equation (3.17),

$$\mathcal{P}_{0,0}(t) = \left(\delta(t) + \mu\mathcal{P}_{1,1}(t)\right) * \sum_{r=0}^{\infty}\left\{e^{-\lambda}\ \frac{\left((\eta+\xi)t\right)^r}{r!} * [\phi_1(t)]^{*r}\right\},$$

$$\mathcal{P}_{n,0}(t) = \psi_n(t) * \mathcal{P}_{0,0}(t),\ n = 1,2,3,\ldots$$

where $\psi_n(t)$for all *n* are derived in the appendix and

$$\mathcal{P}_{n,1}(t) = \int_0^t \sum_{i=1}^{\infty} [\phi \mathcal{P}_{i,0}(y) + \bar{q}\eta \mathcal{P}_{i+1,0}(y)] \beta^{n-i} \left( I_{n-i}\big(\alpha(t-y)\big) - I_{n+i}\big(\alpha(t-y)\big) \right) e^{-(\lambda+\mu)(t-y)} dy, n = 2,3,4, \dots$$

**Remark 1:**

When $q = 1$, equation (3.12) and (3.3) becomes

$$\mathcal{P}_{n,1}(t) = \int_0^t \sum_{i=1}^{\infty} \phi \mathcal{P}_{i,0}(y) \beta^{n-i} \left( I_{n-i}\big(\alpha(t-y)\big) - I_{n+i}\big(\alpha(t-y)\big) \right) e^{-(\lambda+\mu)(t-y)}\, dy,$$

$$n = 2,3,4, \dots$$

and

$$\mathcal{P}_{n,0}(t) = \psi_n(t) * \mathcal{P}_{0,0}(t), n = 1,2,3, \dots$$

which is the same as equation (5.6) of Sudhesh et al [7] with the equivalence$\gamma = \phi,\ \mu_0 = \eta$ and $\mu_1 = \mu$.

## 4. Stationary Probabilities

Denote the steady – state probability of $\mathcal{P}_{n,j}(t)$ by $\pi_{n,j}$. The final value theorem of Laplace transform states$\pi_{n,j} = \lim_{s\to 0} s\hat{P}_{n,j}(s)$. From equation (3.6), we observe that

$$\lim_{s\to 0} s\hat{\mathcal{P}}_{n,0}(s) = \lim_{s\to 0} s\left\{ \left(1 + \mu \hat{\mathcal{P}}_{1,1}(s)\right) \hat{\psi}_n(s) \sum_{r=0}^{\infty} \frac{(\eta+\xi)^r [\hat{\psi}_1(s)]^r}{(s+\lambda)^{r+1}} \right\},$$

and hence

$$\pi_{n,0} = \mu \psi_n \sum_{r=0}^{\infty} \frac{(\eta+\xi)^r [\psi_1]^r}{(\lambda)^{r+1}} \pi_{1,1} = \frac{\mu \psi_n \pi_{1,1}}{\lambda - (\eta+\xi)\psi_1}, n = 1,2,3, \dots \qquad (4.1)$$

where

$$\psi_n = \lim_{s\to 0} \hat{\psi}_n(s)$$

$$= \lim_{s\to 0} \frac{\lambda^n}{\prod_{i=1}^{n}(s + \phi + \eta + i\xi)} \lim_{s\to 0} \frac{{}_1F_1\left(\frac{q\eta}{\xi} + n + 1\,; \frac{s+\phi+\eta}{\xi} + n + 1; -\frac{\lambda}{\xi}\right)}{{}_1F_1\left(\frac{q\eta}{\xi} + n\,; \frac{s+\phi+\eta}{\xi} + n; -\frac{\lambda}{\xi}\right)}$$

$$= \frac{\lambda^n}{\prod_{i=1}^{n}(\phi + \eta + i\xi)} \frac{{}_1F_1\left(\frac{q\eta}{\xi} + n + 1\,; \frac{\phi+\eta}{\xi} + n + 1; -\frac{\lambda}{\xi}\right)}{{}_1F_1\left(\frac{q\eta}{\xi} + n\,; \frac{\phi+\eta}{\xi} + n; -\frac{\lambda}{\xi}\right)}$$

Similarly, from equation (3.5), we get

$$\pi_{0,0} = \frac{\mu \pi_{1,1}}{\lambda - (\eta+\xi)\psi_1}$$

Therefore, equation (4.1) reduces to

$$\pi_{n,0} = \psi_n \pi_{0,0}, \qquad n = 1,2,3, \dots \qquad (4.2)$$

Laplace transform of equation (3.12) results in

$$\hat{\mathcal{P}}_{n,1}(s) = \sum_{i=1}^{\infty} [\phi \hat{\mathcal{P}}_{i,0}(s) + \bar{q}\eta \hat{\mathcal{P}}_{i+1,0}(s)] \beta^{n-i} \left\{ \frac{\left(p - \sqrt{p^2-\alpha^2}\right)^{n-i}}{\alpha^{n-i}\sqrt{p^2-\alpha^2}} - \frac{\left(p - \sqrt{p^2-\alpha^2}\right)^{n+i}}{\alpha^{n+i}\sqrt{p^2-\alpha^2}} \right\},$$

Hence, it is seen that

$$\lim_{s\to 0} s\hat{\mathcal{P}}_{n,1}(s) = \sum_{i=1}^{\infty}\left[\phi \lim_{s\to 0} s\hat{\mathcal{P}}_{i,0}(s) \ + \bar{q}\eta \lim_{s\to 0} s\hat{\mathcal{P}}_{i+1,0}(s)\right]\beta^{n-i}\left\{\frac{\left(p-\sqrt{p^2-\alpha^2}\right)^{n-i}}{\alpha^{n-i}\sqrt{p^2-\alpha^2}} - \frac{\left(p-\sqrt{p^2-\alpha^2}\right)^{n+i}}{\alpha^{n+i}\sqrt{p^2-\alpha^2}}\right\}.$$

On further simplification, we get

$$\pi_{n,1} = \sum_{i=1}^{\infty}\left[\phi\pi_{i,0} + \bar{q}\eta\pi_{i+1,0}\right]\beta^{n-i}\frac{1}{\mu-\lambda}\left\{\left(\frac{2\lambda}{\alpha}\right)^{n-i} - \left(\frac{2\lambda}{\alpha}\right)^{n+i}\right\},$$

Therefore, substituting for $\pi_{i,0}$ and $\pi_{i+1,0}$from equation (5.2) in the above equation yields

$$\pi_{n,1} = \frac{\pi_{0,0}}{\mu-\lambda}\left\{\sum_{i=1}^{\infty}[\phi\psi_i + \bar{q}\eta\psi_{i+1}]\left(\left(\frac{\lambda}{\mu}\right)^{n-i} - \left(\frac{\lambda}{\mu}\right)^{n}\right)\right\}$$
$$= \frac{\pi_{0,0}}{\mu-\lambda}\left(\frac{\lambda}{\mu}\right)^{n}\left\{\sum_{i=1}^{\infty}[\phi\psi_i + \bar{q}\eta\psi_{i+1}]\left(\left(\frac{\mu}{\lambda}\right)^{i} - 1\right)\right\}$$

which can be expressed as

$$\pi_{n,1} = \frac{\pi_{0,0}}{\mu-\lambda}\left(\frac{\lambda}{\mu}\right)^{n} R \qquad (4.3)$$

where

$$R = \sum_{i=1}^{\infty}[\phi\psi_i + \bar{q}\eta\psi_{i+1}]\left(\left(\frac{\mu}{\lambda}\right)^{i} - 1\right).$$

Using the fact that the total probability is always one, we obtain

$$\pi_{0,0} + \sum_{n=1}^{\infty}\pi_{n,0} + \sum_{n=1}^{\infty}\pi_{n,1} = 1,$$

and hence

$$\pi_{0,0} + \pi_{0,0}\sum_{n=1}^{\infty}\psi_n + \pi_{0,0}\sum_{n=1}^{\infty}\frac{R}{\mu-\lambda}\left(\frac{\lambda}{\mu}\right)^{n} = 1.$$

.
Therefore, we get

$$\pi_{0,0}\left[1 + \sum_{n=1}^{\infty}\psi_n + \frac{R\lambda}{(\mu-\lambda)^2}\right] = 1 \text{ where } \lambda < \mu,$$

which in turn gives

$$\pi_{0,0} = \frac{1}{1 + \sum_{n=1}^{\infty}\psi_n + \frac{R\lambda}{(\mu-\lambda)^2}}. \qquad (4.4)$$

The stationary system size probabilities are thus obtained explicitly as given by equations (4.2), (4.3) and (4.4).

## 5. Results of Numerical Study

This section highlights the trend of $P_{n,j}(t)$ versus time for appropriate selection of parameters. Further, the variability of the mean and variance of $N(t)$ against time are also illustrated. To aid numerical computation, the parameter, $n$ is truncated at 25 and the other parameter values are chosen as follows: $\lambda = 3, \mu = 5, q = 0.5, \eta = 2, \phi = 1$ and $\xi = 0.9$.

Figure 2 and Figure 3 portrays the variability of $P_{n,0}(t)$ versus time $t$ for different choices of $n$. Observe that $P_{n,0}(t)$ is equal to 0 initially and thereby increases steadily upto certain point of time and gradually decreases to merge with the respective steady state probabilities.

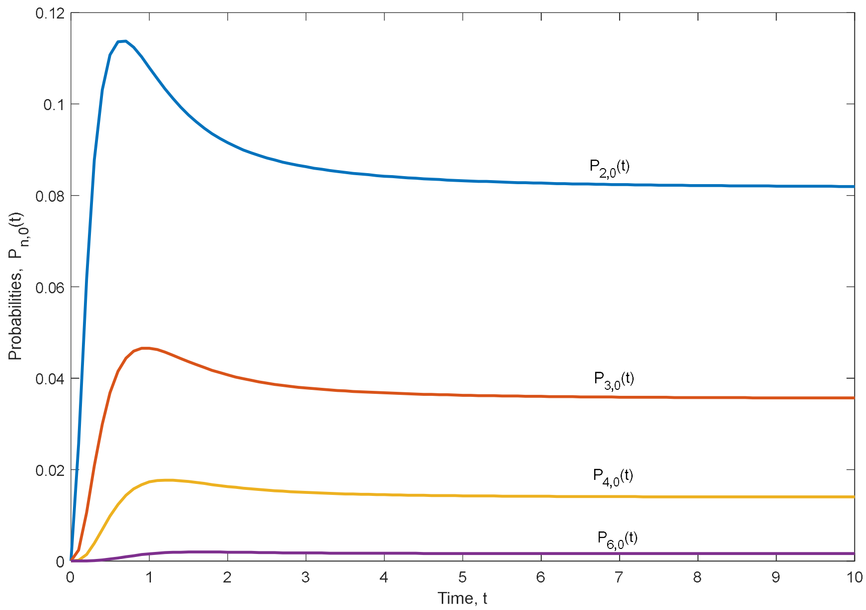


**Figure 2. Variation of $P_{n,0}(t)$ against Time**

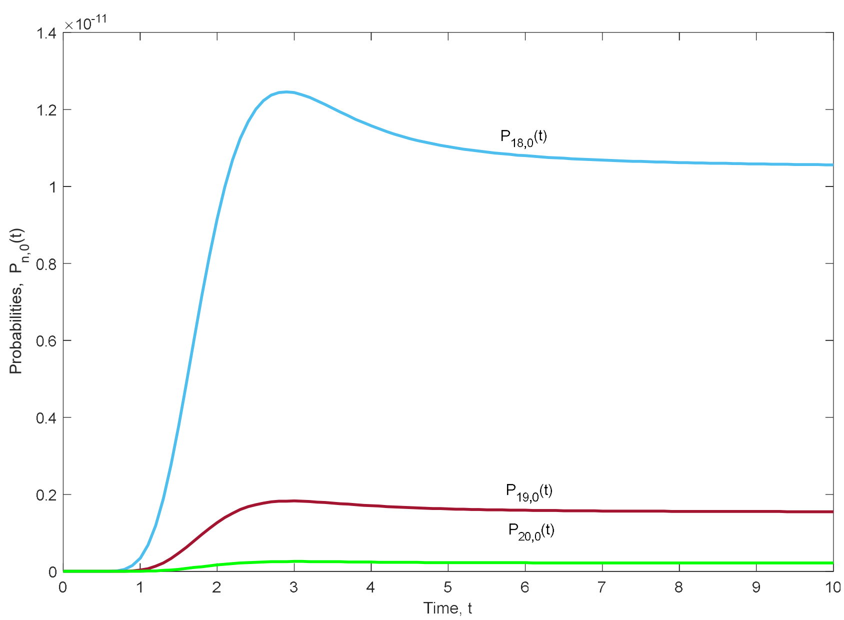


**Figure 3. Variation of $P_{n,0}(t)$ against Time**

Figure 4 and Figure 5 sketches the variability of $P_{n,1}(t)$ against time for different choices of *n* with identical parameters. It is seen that for a specific choice of *n,* the transient state probability increases as time progresses and merges with the respective steady state probabilities. However, for a specific value of *t,* the probability curve descends as the '*n*' value increases.

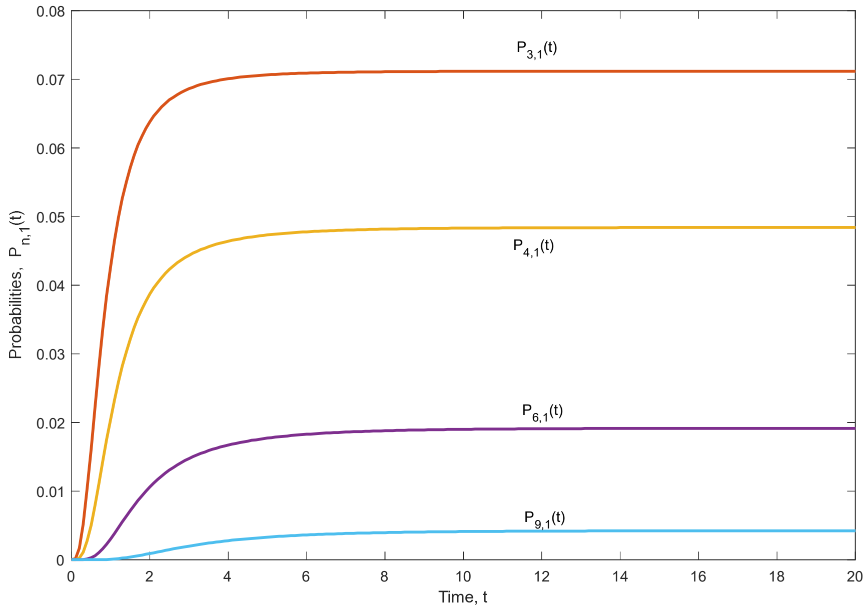


**Figure 4. Variation of $P_{n,1}(t)$ against Time**

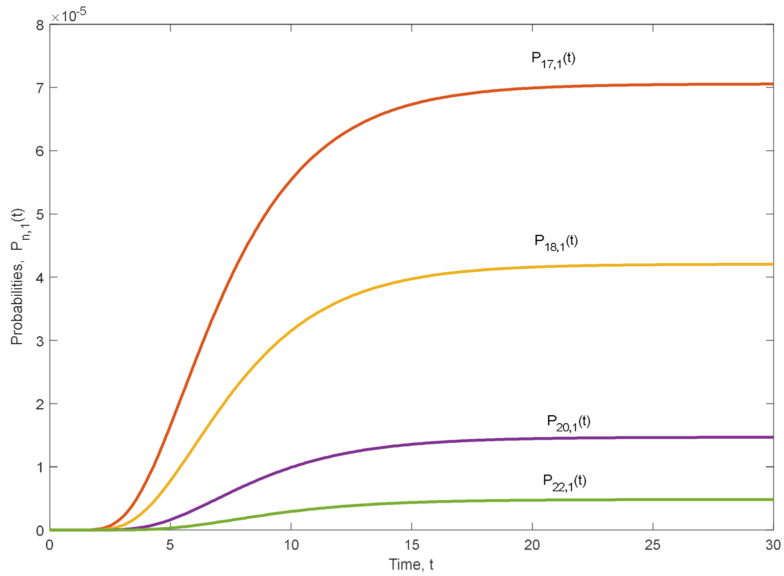


**Figure 5. Variation of $P_{n,1}(t)$ against Time**

Having found all the time dependent probabilities, certain measures of effectiveness will be interesting to the readers. Figure 6 and Figure 7 depicts the effect of the mean and variance of $N(t)$ versus time for different choices of $\xi$. Observe that the mean curve and the variance curve ascends as time increases. For a given time, it is seen that the time-dependent mean as well as variance decrease as the value of $\xi$ increases. This behaviour is justified because when the server's vacation duration is more, the absence of the server for a extended duration will lead to an increase of the average number in the system and hence also the variance.

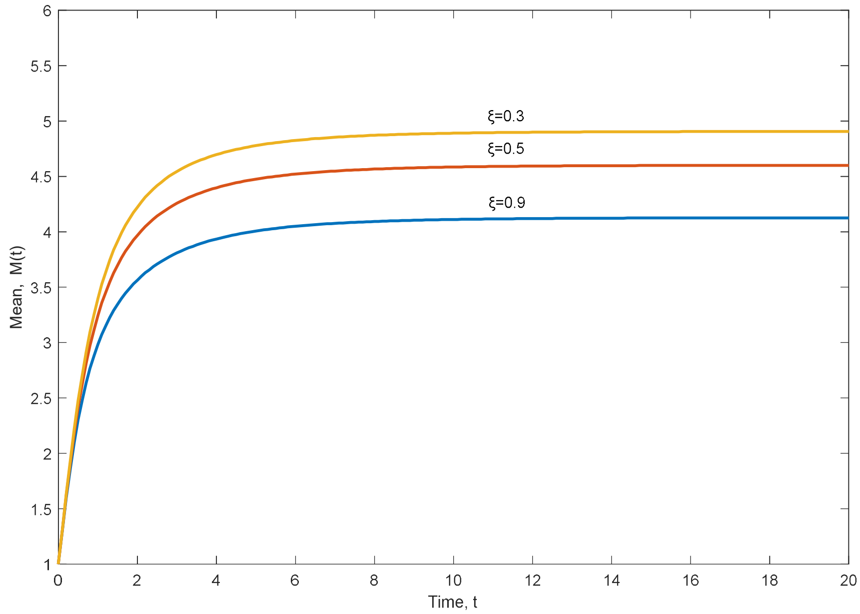


**Figure 6. Variability of $M(t)$for different choices of $\xi$.**

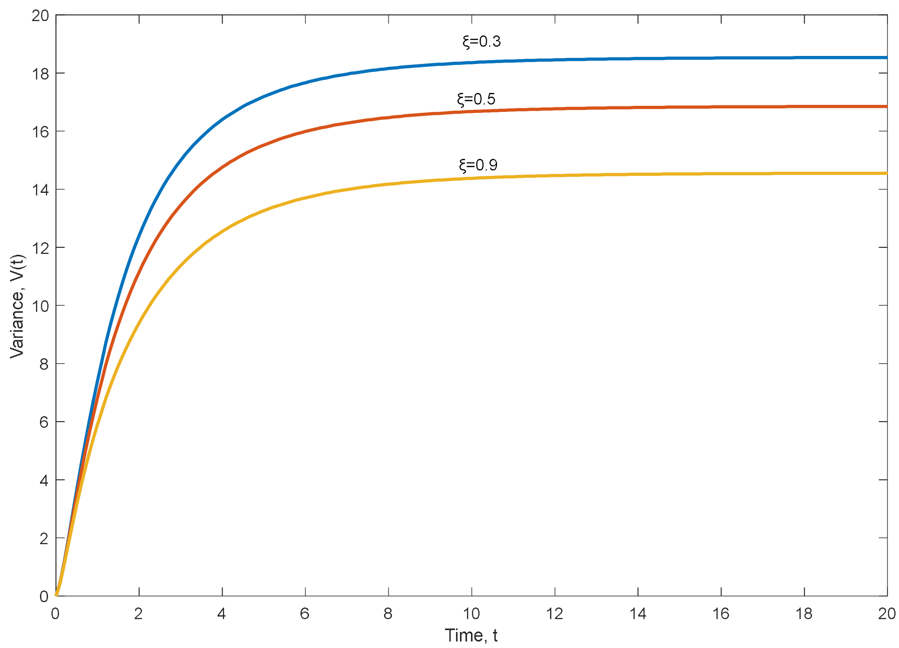


**Figure 7. Variability of $V(t)$for different choices of $\xi$.**

## 6. Conclusion

Queueing models subject to various vacation strategies have wide applications to diverse fields with a specific emphasis on computer and communication systems wherein the single main server provides a variety of service and in case of failure, will be alternated by a backup server. There is ample research addressing the system size probabilities and related measures of effectiveness for vacation queueing models and their variants in the literature. However, most of the analysis pertains to the stationary regime, which poses limitations to further analysis. This article addresses the transient analysis of an *M/M/1* vacation queue with interruption. Closed form analytical expressions for the time-dependent state probabilities are obtained using continued fraction and generating function methodologies. The study can be extrapolated to multi-server models to cover broad applications.

## Appendix

**Derivation of $\psi_n(t)$**

The confluent hypergeometric function represented by $1F1(a;c;z)$ has a series representation given by

$${}_1F_1(a;c;z)=1+\frac{a}{c}\frac{z}{1!}+\frac{a}{c}\frac{a(a+1)}{(c+1)}\frac{z^2}{2!}+\cdots=1+\sum_{k=1}^{\infty}\frac{\prod_{j=1}^{k-1}(a+i)}{\prod_{i=0}^{k-1}(c+i)}\frac{z^k}{k!}$$

Consider the expression for $\hat{\psi}_n(s)$ obtained as

$$\hat{\psi}_n(s)=\left(\frac{\lambda}{\xi}\right)^n\frac{1}{\prod_{i=1}^{n}\left(\frac{s+\phi+\eta}{\xi}+i\right)}\frac{1F1\left(\frac{q\eta}{\xi}+n+1\,;\frac{s+\phi+\eta}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{1F1\left(\frac{q\eta}{\xi}+1\,;\frac{s+\phi+\eta}{\xi}+1;-\frac{\lambda}{\xi}\right)}\qquad(A.1)$$

where

$${}_1F_1\left(\frac{q\eta}{\xi}+n+1\,;\frac{s+\phi+\eta}{\xi}+n+1;-\frac{\lambda}{\xi}\right)=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{k}(q\eta+(n+j)\xi)}{\prod_{i=1}^{n+k}(s+\phi+\eta+i\xi)}\frac{(-\lambda)^k}{\xi^{k-n}k!}$$

and hence

$$\frac{{}_1F_1\left(\frac{q\eta}{\xi}+n+1\,;\frac{s+\phi+\eta}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{\prod_{i=1}^{n}\left(\frac{s+\phi+\eta}{\xi}+i\right)}=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{k}(q\eta+(n+j)\xi)}{\prod_{i=1}^{n+k}(s+\phi+\eta+i\xi)}\frac{(-\lambda)^k}{\xi^{k-n}k!}$$

Using partial fractions to the product terms in the denominator of the above equation results in

$$\frac{{}_1F_1\left(\frac{q\eta}{\xi}+n+1\,;\frac{s+\phi+\eta}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{\prod_{i=1}^{n}\left(\frac{s+\phi+\eta}{\xi}+i\right)}=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{k}(q\eta+(n+j)\xi)}{k!}\frac{(-\lambda)^k}{\xi^{2k-1}}\sum_{i=1}^{n+k}\left(\frac{(-1)^{i-1}}{(n+k-i)!\,(i-1)!}\right)\left(\frac{1}{s+\phi+\eta+i\xi}\right)\qquad(A.2)$$

Now, consider the term in the denominator of $\hat{\psi}_n(s)$ as

$${}_1F_1\left(\frac{q\eta}{\xi}+1\,;\frac{s+\phi+\eta}{\xi}+1;-\frac{\lambda}{\xi}\right)=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{k}(q\eta+j\xi)}{\prod_{i=1}^{k}(s+\phi+\eta+i\xi)}\frac{(-\lambda)^k}{\xi^k k!}=\sum_{k=0}^{\infty}(-\lambda)^k\hat{a}_k(s)$$

where $\hat{a}_k(s) = \frac{\prod_{j=1}^{k}(q\eta + j\xi)}{\prod_{i=1}^{k}(s + \phi + \eta + i\xi)}\left(\frac{1}{\xi^k k!}\right)$ and $\hat{a}_0(s) = 1$. Resolving the product terms in the denominator using partial fractions leads to

$$\hat{a}_k(s) = \frac{1}{\xi^{2k-1}k!}\sum_{r=1}^{k}\frac{\prod_{j=1}^{k}(q\eta + j\xi)(-1)^{r-1}}{(k-r)!\,(r-1)!}\frac{1}{s + \phi + \eta + r\xi}, \qquad \text{for } k = 1,2,3 \ldots$$

From the identity mentioned in ([3]), it is seen that

$$\left[{}_1\mathrm{F}_1\left(\frac{q\eta}{\xi} + 1\,; \frac{s + \phi + \eta}{\xi} + 1; -\frac{\lambda}{\xi}\right)\right]^{-1} = \left[\sum_{k=0}^{\infty}\hat{a}_k(s)\,(-\lambda)^k\right]^{-1} = \sum_{k=0}^{\infty}\hat{d}_k(s)\,\lambda^k \qquad (A.3)$$

where $\hat{d}_0(s) = 1$ and

$$\hat{b}_k(s) = \begin{vmatrix} \hat{a}_1(s) & 1 & & \ldots & & \\ \hat{a}_2(s) & \hat{a}_1(s) & 1 & \ldots & & \\ \hat{a}_3(s) & \hat{a}_2(s) & \hat{a}_2(s) & \ldots & & \\ \ldots. & \ldots. & \ldots & \ldots & \ldots & \ldots \\ \hat{a}_{k-1}(s) & \hat{a}_{k-2}(s) & \hat{a}_{k-3}(s) & \ldots & \hat{a}_1(s) & 1 \\ \hat{a}_k(s) & \hat{a}_{k-1}(s) & \hat{a}_{k-2}(s) & \ldots & \hat{a}_2(s) & \hat{a}_1(s) \end{vmatrix}$$

$$= \sum_{i=1}^{k}(-1)^{i-1}\hat{a}_i(s)\hat{d}_{k-i}(s), \qquad k = 1,2,3,\ldots.$$

Substituting equation (A.3) and equation (A.2) in equation (A.1), we get

$$\hat{\psi}_n(s) = \xi^n\lambda^n\sum_{k=0}^{\infty}(-\lambda)^k\frac{\prod_{j=1}^{k}(q\eta + (n+j)\xi)}{\prod_{j=1}^{k}(q\eta + j\xi)}\hat{a}_{n+k}(s)\sum_{j=1}^{\infty}\lambda^j\hat{d}_j(s)\,.$$

Inverse Laplace transform of $\hat{\psi}_n(s)$ leads to

$$\psi_n(t) = \lambda^n\xi^n\sum_{k=0}^{\infty}(-\lambda)^k\frac{\prod_{j=1}^{k}(q\eta + (n+j)\xi)}{\prod_{j=1}^{k}(q\eta + j\xi)}a_{n+k}(t) * \sum_{j=1}^{\infty}\lambda^j d_j(t)\,,$$

where

$$a_k(t) = \frac{1}{\xi^{2k-1}k!}\sum_{r=1}^{k}\frac{\prod_{j=1}^{k}(q\eta + j\xi)(-1)^{r-1}}{(r-1)!\,(k-r)!}e^{-(\phi + q\eta + r\ \ )t}, k = 1,2,\ldots$$

and

$$d_k(t) = \sum_{i=1}^{k}(-1)^{i-1}a_i(t) * d_{k-i}(t), k = 2,3,\ldots, \quad d_1(t) = a_1(t)$$

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